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Optimal Design of Frame Structures Using Multivariate Spline Approximation

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The objective of this work is to demonstrate the efficiency and applicability of the optimization algorithm using a multivariate spline approximation. In the present algorithm, based on the function values and first-order derivatives of the constraints available at the intermediate points of optimization, an explicit approximation of constraint functions is created by using the least squares spline algorithm. The nonlinearity of the function is adaptively updated using feedback information from the previous two iterations for finding the order of spline approximation. In addition, constraint deletion and design variable linking concepts are employed in solving the approximate problem by using the quasianalytical sequential quadratic programming and dual methods. The behavior constraints include stresses, displacements, and local buckling in the optimum design of frame structures. To demonstrate the broad applicability of the optimization algorithm, the cross-sectional dimensions are directly selected as the design variables for frame problems having thin-walled rectangular and tube cross-sectional members, and the cross-sectional areas are selected as design variables for an *I*-section problem.

Introduction

THE optimal design of frame structures is much more complicated than that of truss and membrane structures. For frame design problems, each element usually includes multiple design variables, and the element stiffness matrix has a nonlinear relationship with the element cross-sectional dimensions (CSDs). As a result, the optimization methods that can efficiently solve truss and membrane problems are generally inadequate for solving frame problems. In the past 10 years, the optimum design of frame structures has been emphasized, and constraint approximations and algorithms have been developed significantly. The main methods have followed several paths. The first and most common method is oriented toward civil engineering applications. Since many of the structures in such applications are built using standard section members (e.g., wide flange *I* beam), it has become popular to use assumed size-inertia relationships of the form

$$I = CA^p \tag{1}$$

where I is the cross-sectional moment of inertia, C and p are constants, and A is the cross-sectional area.[1–4] This relationship represents an assumption governing the geometry of the cross section during redesign. This approach has the advantage of representing the design of a structural element with only one design variable, consequently leading to optimum design problems with relatively few design variables, but it restricts the freedom of the design problem. Although a lack of design freedom is not a serious disadvantage in many civil structure designs, it can be a severe limitation for weight-critical design applications, such as used in the aerospace and automotive industries.

The restrictions of Eq. (1) can be alleviated by selecting the cross-sectional dimensions as the design variables.[5,6] However, for a general multivariable design element, the behavior constraint approximations based on the first-order Taylor series expansion in terms of the element CSDs or their reciprocals are usually less accurate because of the highly nonlinear nature in the CSDs design space. Therefore, this method usually requires a relatively large number of iterations and may also exhibit convergence difficulties.

A widely used approach to improve the efficiency of frame optimization is to select intermediate design variables so that accurate approximations for the structural behavior constraints can be generated.[7–10] In this method, it is important to carefully select the intermediate variables. In Refs. 7–9, the reciprocal section properties (RSP) are selected as the intermediate design variables, which are defined as follows:

$$\begin{aligned} X_{i1} &= 1/A_i, \qquad X_{i2} = 1/J_i \\ X_{i3} &= 1/I_{yi}, \qquad X_{i4} = 1/I_{zi} \end{aligned} \tag{2}$$

where X_{ij} is the jth reciprocal variable of the ith element, A_i the cross-sectional area, J_i the polar moment of inertia, I_{yi} the moment of inertia about the y axis, and I_{zi} the moments of inertia about the z axis for the ith element. For some frame design problems, particularly for problems with displacement constraints, the approximate constraints obtained by selecting the intermediate design variables stated in Eq. (2) and expansions using the Taylor series are closer to the actual constraints. However, for frame design problems with arbitrary constraints, selecting the intermediate variables is generally difficult and needs experience and knowledge.

In this paper, an optimization algorithm using a multivariate spline approximation is applied in solving the optimal design of frame structures. The approximate constraint functions are formed by using the least squares spline algorithm based on multiple design points instead of using the first-order Taylor series expansion based on a single design point, and the nonlinearity of the function is adaptively incorporated through feedback information from the previous iterations. Therefore, this algorithm has a more accurate nonlinear approximation than those algorithms using only single-point information and fixed-approximate models such as direct or reciprocal design variables. More accurate approximations and dual methods of solution provide increased efficiency for structural optimization problems, particularly for frame designs having highly nonlinear constraints and a large number of design variables. Several frame problems were used to examine the efficiency and robustness of the present method. The cross-sectional dimensions were directly selected as the design variables for frame problems having thin-walled rectangular and tube cross-sectional members, and the cross-sectional areas were selected as the design variables for an *I*-section problem to demonstrate the broad applicability of the optimization algorithm.

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Optimization Algorithm

The structural optimization problem is stated as

$$\text{minimize:} \quad f(X) \tag{3a}$$

$$\text{subject to:} \quad g_j(X) \le 0, \qquad (j = 1, 2, \dots, J) \tag{3b}$$

$$x_i^L \le x_i \le x_i^U, \qquad (i = 1, 2, \dots, N) \tag{3c}$$

where X is a vector of design variables, $f(X)$ is the objective function, $g_j(X)$ is the jth behavior constraint, x_i^L and x_i^U are the lower and upper limits on the ith design variable, respectively, and J and N denote the number of behavior constraints and design variables, respectively.

For a large-scale structure optimization or a frame design with high nonlinearity and a large number of design variables, an effective way to improve the computational efficiency is to change the original problem of Eq. (3) into a sequence of explicit approximate problems by using approximation concepts including constraint deletion and design variable linking, as well as the Taylor series expansion. In general, the approximate problems are formed by using the first-order Taylor series expansion in terms of direct and reciprocal design variables. For complex nonlinear problems, some researchers have used higher order approximations.[11,12] To reduce the expense of calculating second-order derivatives for displacement and stress constraints, Ref. 12 simplified the Hessian matrices to diagonal matrices by invoking Saint-Venant's principle and assuming that changes in the property of an element would affect only the stress in that element.

Both the first-order and higher order Taylor approximations are based on a single design point. As the structure is being resized, new approximations are constructed at new design points. In this approach, previous analyses' information is discarded and not used to improve the later approximations. In a survey paper,[13] several recent developments on multipoint constraint approximations in structural optimization are summarized. An approximation using two and three points was proposed in Ref. 14, and the authors indicated that the approximation was good when it represented an interpolation, whereas the improvement in accuracy was marginal when it represented an extrapolation. Another two-point exponential approximation was proposed in Ref. 15, which is a linear Taylor approximation in terms of intervening variables $y_i = x_i^{p_i}$ ($i = 1, 2, \dots, N$) and the exponent p_i for each design variable was determined by matching the derivatives of the previous design point. To avoid large exponentials, the value of p_i was limited to +1 or −1. Also, because of the logarithmic term in computing p_i, some checks were needed to avoid computational problems.

In this paper, an optimization algorithm based on a multivariate spline approximation is presented, in which the previous information is used in constructing a least squares spline in N dimensions for evaluating the approximate function values. The main procedure includes the following.

1) Construct explicit spline approximate functions for active constraints by using the least squares spline algorithm based on the function values and first-order derivatives of the constraints available at the intermediate points of optimization. The nonlinearity of the function is adaptively controlled by the feedback information from the previous iterations.

2) Formulate a sequential quadratic programming (SQP) problem by expanding the splines as the first-order Taylor series for the constraints and as the second-order Taylor series for the objective function.

3) Find the optimum solution of the SQP problem by using the dual method, in which only spline approximations are used to obtain the function values and derivatives of constraint functions in each iteration instead of exact finite element method (FEM) and sensitivity analyses.

4) Find the optimum solution of the original problem iteratively, in which exact finite element analysis is used for improving the spline approximations in each iteration. The details on the present algorithm are described in the following.

Multivariate Spline Algorithm

Assuming the coordinates of $M + 1$ previous design points as $(x_{1,0}, x_{2,0}, \dots, x_{N,0}, Z_0)$, $(x_{1,1}, x_{2,1}, \dots, x_{N,1}, Z_1)$, $\dots$, and $(x_{1,M}, x_{2,M}, \dots, x_{N,M}, Z_M)$, and selecting the number of spline knots as $m + 1$, the interval $[a_j, b_j]$ ($j = 1, 2, \dots, N$) is divided as

$$\begin{aligned} a_1 &= \bar{x}_{1,0} < \bar{x}_{1,1} < , \dots, < \bar{x}_{1,m} = b_1 \\ a_2 &= \bar{x}_{2,0} < \bar{x}_{2,1} < , \dots, < \bar{x}_{2,m} = b_2 \\ &\dots\dots \\ a_N &= \bar{x}_{N,0} < \bar{x}_{N,1} < , \dots, < \bar{x}_{N,m} = b_N \end{aligned} \tag{4}$$

By extending these knot sequences on the left and right limits,

$$\begin{aligned} &\bar{x}_{1,-n} < , \dots, \bar{x}_{1,-1} < a_1 = \bar{x}_{1,0} < \bar{x}_{1,1} < , \dots, < \bar{x}_{1,m} \\ &\quad = b_1 < \bar{x}_{1,m+1} < , \dots, < \bar{x}_{1,m+n} \\ &\bar{x}_{2,-n} < , \dots, \bar{x}_{2,-1} < a_2 = \bar{x}_{2,0} < \bar{x}_{2,1} < , \dots, < \bar{x}_{2,m} \\ &\quad = b_2 < \bar{x}_{2,m+1} < , \dots, < \bar{x}_{2,m+n} \\ &\dots\dots \\ &\bar{x}_{N,-n} < , \dots, < \bar{x}_{N,-1} < a_N = \bar{x}_{N,0} < \bar{x}_{N,1} < , \dots, < \bar{x}_{N,m} \\ &\quad = b_N < \bar{x}_{N,m+1} < , \dots, < \bar{x}_{N,m+n} \end{aligned} \tag{5}$$

where m may not be the same as M, which can be fixed or adjusted automatically by the algorithm.

Based on the recurrence relation of the univariate B spline,[16] a recurrence formula of the multivariate B spline is approximately written as

$$\begin{aligned} B_{i,n+1}(X) &= \frac{(X - \bar{X}_i)^T [D]_i (\bar{X}_{i+n} - \bar{X}_i)}{(\bar{X}_{i+n} - \bar{X}_i)^T (\bar{X}_{i+n} - \bar{X}_i)} B_{i,n}(X) \\ &+ \frac{(\bar{X}_{i+n+1} - X)^T [D]_{i+1} (\bar{X}_{i+n+1} - \bar{X}_{i+1})}{(\bar{X}_{i+n+1} - \bar{X}_{i+1})^T (\bar{X}_{i+n+1} - \bar{X}_{i+1})} B_{i+1,n}(X) \end{aligned} \tag{6}$$

where $[D]_i$ is a diagonal matrix and its elements d_j are

$$d_j = \begin{cases} 1 & \text{if} \quad x_j \in [\bar{x}_{j,i}, \bar{x}_{j,i+1}) \\ 0 & \text{if} \quad x_j \notin [\bar{x}_{j,i}, \bar{x}_{j,i+1}) \end{cases} \qquad j = 1, 2, \dots, N \tag{7}$$

B splines of order one are defined as

$$B_{i,1}(X) = \begin{cases} 0 & \text{if} \quad x_j \notin [\bar{x}_{j,i}, \bar{x}_{j,i+1}) \quad \text{for all } j = 1, 2, \dots, N \\ 1 & \text{otherwise} \end{cases} \tag{8}$$

Based on the B spline of Eq. (6) and the interval divisions of Eq. (5), a multivariate spline function of $n + 1$ order is constructed as

$$S_n(X) = \sum_{k=-n}^{m-1} C_k B_{k,n+1}(X) \tag{9}$$

The corresponding derivatives equations are

$$\frac{\partial S_n(X)}{\partial x_j} = \sum_{k=-n}^{m-1} C_k \frac{\partial [B_{k,n+1}(X)]}{\partial x_j}, \qquad j = 1, 2, \dots, N \tag{10}$$

in which $\partial [B_{k,n+1}(X)] / \partial x_j$ can be obtained by differentiating Eq. (6).

To determine the $m+n$ unknown coefficients C_k ($k=-n,\ldots,m-1$) in Eqs. (9) and (10), a suboptimization problem is constructed as

$$\text{minimize} \qquad G(\boldsymbol{C}) \tag{11a}$$

$$\text{subject to} \qquad E(\boldsymbol{C}) \le \varepsilon \tag{11b}$$

in which ε is an acceptable error value, and

$$G(C) = \sum_{j=1}^{N} \sum_{l=0}^{M} W_{j,l}\left[Z_{j,l} - \frac{\partial S_n(\boldsymbol{X}_l)}{\partial x_j}\right]^2 \tag{12a}$$

$$E(\boldsymbol{C}) = \sum_{l=0}^{M} W_l\,[Z_l - S_n(\boldsymbol{X}_l)]^2 \tag{12b}$$

in which Z_l are function values of the known points, $Z_{j,l}$ ($j = 1, 2, \ldots, N$) are the known derivatives of constraint function with respect to design variable x_j, W_l and $W_{j,l}$ are weighted constants,

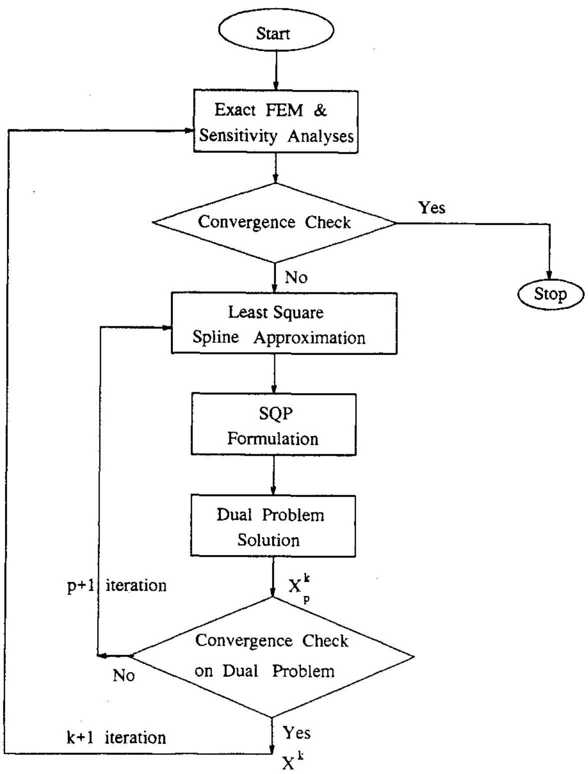

Fig. 1 Optimization algorithm flow chart.

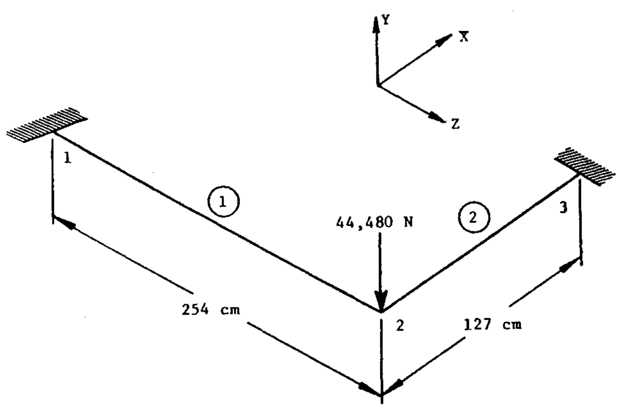

Fig. 2 Two member frame.

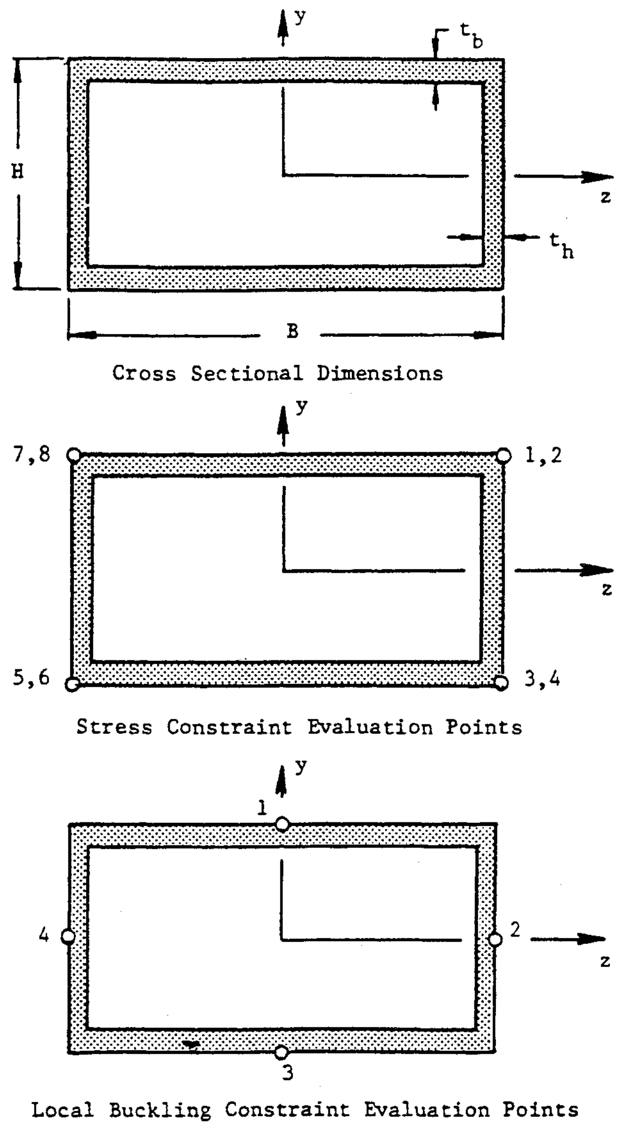

Fig. 3 Thin-walled rectangular cross section.

and the optimization problem is to minimize the difference of exact and approximate function derivatives at the known points and make the values of the approximate function approach the exact values at the known points as close as possible.

This suboptimization problem is solved by using a Lagrange multipliers algorithm. After the optimum solutions C_k ($k=-n,\ldots, m-1$) are determined, the spline function $S_n(\boldsymbol{X})$ given in Eqs. (9) and (10) is constructed.

Spline-Order Calculation

The order $n + 1$ of the spline function has to be determined before constructing the approximate function. The polynomial degree n of the spline functions $S_n(\boldsymbol{X})$ is determined by using a feedback formula. To establish this feedback formula, assuming the intervening variables as $\boldsymbol{S} = (s_1, s_2, \ldots, s_N)^T$, let

$$s_i = x_i^r, \qquad i = 1, 2, \ldots, N \tag{13}$$

where r represents the nonlinearity index of the constraint function. The first-order Taylor's series expansion of the constraint function at the current point $\boldsymbol{X}_k$ in terms of the original variables x_i is given as

$$g(X) = g(X_k) + \frac{1}{r}\sum_{i=1}^{N} x_{i,k}^{(1-r)} \frac{\partial g(X_k)}{\partial x_i}(x_i^r - x_{i,k}^r) \tag{14}$$

Table 1 Final design of two member frame

Member no.	Variables	Initial design	Final design Ref. 7	Ref. 8, run 1	Ref. 8, run 2	Ref. 8, run 5	Ref. 8, run 7	Present paper
1	B	15.20	6.35	6.35	6.35	6.35	6.35	6.35
	H	10.20	6.35	6.35	6.35	6.35	6.35	6.35
	t_b	2.03	0.229	0.229	0.229	0.229	0.229	0.229
	t_h	2.29	0.254	0.254	0.254	0.254	0.254	0.254
2	B	22.90	6.35	18.41	18.36	9.48	18.35	18.36
	H	20.30	25.40	25.40	25.40	25.40	25.40	25.40
	t_b	2.03	1.14	0.348	0.348	0.715	0.349	0.348
	t_h	2.29	0.254	0.254	0.254	0.254	0.254	0.254
Weight, kg		1220.78	133.70	130.82	130.60	132.06	130.74	130.64
No. of analyses		—	21	13	13	11	7	6

Table 2 Iteration history of two member frame

Step no.	Ref. 8, run 1	Ref. 8, run 2	Ref. 8, run 5	Ref. 8, run 7	Present paper
1	1220.78	1220.78	1220.78	1220.78	1220.78
2	531.31	531.31	686.69	313.48	124.86
3	313.48	313.48	397.37	165.10	134.68
4	204.30	204.30	298.48	149.55	131.64
5	182.90	182.05	247.78	135.39	130.74
6	159.41	159.55	204.49	130.35	130.64
7	153.69	151.42	168.88	130.75	
8	145.53	143.72	142.71		
9	138.31	137.09	132.08		
10	132.27	132.49	132.06		
11	130.22	130.59	132.06		
12	130.79	130.60			
13	130.83	130.60			

Table 3 Iteration convergence details of two member frame

Iteration no.	No. of FEM analysis	Spline approx. for SQP formulation	Weight, kg	Max constraint violation
1	1	1	1220.7800	-7.94×10^{-1}
2	1	1	135.6268	-6.99×10^{-4}
		2	127.8315	-2.10×10^{-6}
		3	125.8073	-6.85×10^{-4}
		4	124.9336	-6.82×10^{-5}
		5	124.8663	-7.76×10^{-4}
		6	124.8651	-6.90×10^{-6}
3	1	1	124.8651	4.91×10^{-1}
		2	136.5686	3.17×10^{-2}
		3	134.7226	-1.85×10^{-3}
		4	134.7435	-1.18×10^{-3}
		5	134.6866	-4.51×10^{-5}
		6	134.6870	-5.59×10^{-5}
4	1	1	134.6870	-4.39×10^{-2}
		2	131.6635	-3.73×10^{-4}
		3	131.6631	-3.66×10^{-4}
		4	131.6463	-4.89×10^{-5}
		5	131.6460	-8.04×10^{-5}
5	1	1	131.6460	-9.21×10^{-4}
		2	130.7421	-3.78×10^{-4}
		3	130.7409	-4.59×10^{-4}
6	1	1	130.7409	-5.33×10^{-3}
		2	130.6407	-1.42×10^{-5}
		3	130.6401	-1.96×10^{-5}

This approximation utilized the constraint value and gradient information at X_k. Using this approximation, the function value at X_{k-1} is computed and compared with the exact $g(X_{k-1})$. The r value is calculated such that the difference in exact and approximate $g(X)$ at X_{k-1} goes to zero.

$$g(X_{k-1}) - \left[g(X_k) + \frac{1}{r} \sum_{i=1}^{N} x_{i,k}^{(1-r)} \frac{\partial g(X_k)}{\partial x_i} (x_{i,k-1}^r - x_{i,k}^r) \right] = 0 \tag{15}$$

$$n = |r| \tag{16}$$

where n is the nearest integer of r. It implies that when r is negative, reciprocal design variables are used in the spline approximation of Eq. (9).

Approximate Optimization Problem

Based on the preceding approximate function $S_n(X)$, the original optimization problem of Eq. (3) is changed into a sequence of explicit approximate problems as follows:

$$\text{minimize:} \quad f^{(k)}(X) = f(X) \tag{17a}$$

$$\text{subject to:} \quad g_j^{(k)}(X) = S_n(X) \le 0, \qquad j = 1, 2, \ldots, \bar{J} \tag{17b}$$

$$x_i^L \le x_i \le x_i^U, \qquad i = 1, 2, \ldots, \bar{N} \tag{17c}$$

where k is the iteration number, $\bar{J}$ is the number of active constraints selected by the constraint tolerances, and $\bar{N}$ is the number of linked design variables.

There are many optimization algorithms that can solve the problem (17), such as the feasible directions method, penalty function method, etc. In general, there are fewer active constraints than the design variables for a large-scale structure. Hence, the dual method seems attractive to solve the optimization problem by replacing the minimization of an objective function in the original space consisting of a large number of design variables with a maximization of a dual function in the dual variables space. However, this method requires that the variables in the objective and constraint functions be separated. Although the approximate constraint function of Eq. (17b) is an explicit function, it is still a complex nonlinear function, and it is difficult to separate. A quasianalytical sequential quadratic programming method can be used to solve the problem; that is, the constraint functions of Eq. (17b) can be approximated again by using the first-order Taylor series expansion, and the objective function of Eq. (17a) can be approximated by using the second-order Taylor series. This approximate problem can be described in terms of reciprocal variables $[y_i = (1/x_i)]$

$$\text{minimize:} \quad \bar{f}(Y) = f(Y_p) + \nabla f(Y_p)^T (Y - Y_p) + (1/2)(Y - Y_p)^T H_f(Y_p)(Y - Y_p) \tag{18a}$$

$$\text{subject to:} \quad \bar{g}_j(Y) = g_j^{(k)}(Y_p) + \nabla g_j^{(k)}(Y_p)^T (Y - Y_p) \le 0 \tag{18b}$$

$$j = 1, 2, \ldots, \bar{J}$$

$$y_i^L \le y_i \le y_i^U, \qquad i = 1, 2, \ldots, \bar{N} \tag{18c}$$

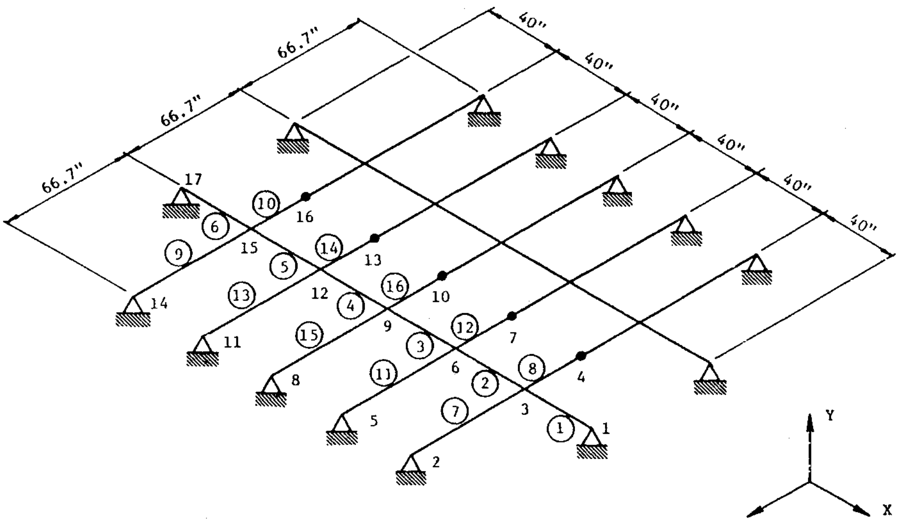

Fig. 4 Grillage structure, 2 × 5.

Table 4 Final design for 2 × 5 grillage frame

Linking group	Member no.	Variables	Initial design	Final design Ref. 7	Ref. 8, run 1	Ref. 8, run 2	Ref. 8, run 5	Ref. 8, run 7	Present paper
		B	12.00	6.10	9.86	10.60	3.26	4.27	3.45
1	1–6	H	15.00	20.00	20.00	19.99	20.00	20.00	20.00
		t_b	0.95	0.159	0.096	0.105	0.224	0.234	0.204
		t_h	0.80	0.093	0.091	0.090	0.096	0.094	0.092
		B	12.00	8.28	11.29	11.59	1.91	1.84	11.86
2	7–10	H	15.00	15.20	19.99	19.98	9.07	9.14	19.03
		t_b	0.95	0.074	0.093	0.095	0.148	0.131	0.132
		t_h	0.80	0.064	0.074	0.074	0.055	0.056	0.055
		B	12.00	6.33	15.94	16.85	12.01	11.64	11.25
3	11–14	H	15.00	20.00	20.00	20.00	20.00	20.00	20.00
		t_b	0.95	1.00	0.325	0.299	0.595	0.514	0.512
		t_h	0.80	0.098	0.095	0.094	0.103	0.098	0.097
		B	12.00	11.50	17.62	18.81	14.20	17.65	15.25
4	15–16	H	15.00	20.00	19.99	19.99	20.00	20.00	20.00
		t_b	0.95	1.00	0.657	0.614	0.683	0.643	0.592
		t_h	0.80	0.117	0.119	0.119	0.113	0.118	0.117
Volume, in.^{3}			32,393.4	7927.00	7887.6	7913.5	7546.6	7505.3	7501.1
No. of analyses			—	41	14	15	14	11	9

Table 5 Iteration history for 2 × 5 grillage frame

Step no.	Ref. 8, run 1	Ref. 8, run 2	Ref. 8, run 3	Ref. 8, run 4	Present paper
1	32,382.4	32,382.4	32,382.4	32,382.4	32,393.4
2	25,259.7	25,268.5	21,663.8	20,339.4	17,446.8
3	19,997.0	20,724.0	13,969.4	14,469.9	10,623.5
4	17,270.7	17,578.1	11,573.5	11,099.0	9001.1
5	14,898.8	14,605.3	9712.1	8762.0	8093.2
6	12,535.6	12,916.8	8644.9	7544.4	7531.6
7	11,092.8	11,463.6	7807.2	7523.9	7509.6
8	10,032.6	10,630.3	7730.0	7526.1	7501.3
9	9156.0	9115.1	7717.4	7525.9	7501.1
10	8414.0	8753.1	7641.5	7510.4	
11	7881.9	7959.9	7581.5	7505.2	
12	7886.5	7925.6	7557.1		
13	7884.7	7914.4	7550.4		
14	7887.6	7917.5	7545.6		
15		7913.5			

where p represents the pth iteration of the SQP, $\nabla f(Y_p)$ and $H_f(Y_p)$ are the gradient vector and Hessian matrix of the objective function at Y_p, respectively, and $g_j(Y_p)$ and $\nabla g_j(Y_p)$ are the function and gradient of the jth constraint function at Y_p, respectively, obtained directly by computing the spline function instead of the finite element analysis.

The Lagrange function of the optimization problem of Eq. (18) can be defined as

$$L(Y, \lambda) = \bar{f}(Y) + \sum_{j=1}^{\bar{J}} \lambda_j g_j(Y) \tag{19}$$

Application of the Kuhn-Tucker conditions to this convex and separable approximate subproblem results in a min-max optimization problem; that is, the dual problem of Eq. (18) can be written as

$$\max_{\lambda \geq 0} l(\lambda) \tag{20}$$

$$l(\lambda) = \min_{Y^L \leq Y \leq Y^U} L(Y, \lambda) \tag{21}$$

where $l(\lambda)$ is defined as the dual function. According to dual theory, Y^* and λ^* will be the solution of problem (18) and dual problem (20), respectively, if $\bar{f}(Y^*)$ is equal to $l(\lambda^*)$. The iteration formulas of design variables can be obtained by substituting Eq. (18) into Eqs. (19–21)

$$Y_{p+1} = Y_p - H_f(Y_p)^{-1} [\nabla G^{(k)}(Y_p)\lambda_p + \nabla f(Y_p)] \tag{22}$$

Table 6 Final design for helicopter tail boom

				Final design					
Linking group	Member no.	Vari-ables	Initial design	Ref. 6	Ref. 8, run 1	Ref. 8, run 3	Ref. 8, run 4	Ref. 8, run 5	Present paper
1	1–4	*R*	2.0	2.6695	3.1432	3.1473	3.0816	3.1198	3.1680
		t	0.051	0.0880	0.0801	0.0811	0.0782	0.0794	0.0808
2	5–8	*R*	2.0	1.9152	1.2850	1.4169	1.3848	1.3364	1.0502
		t	0.051	0.0487	0.0355	0.0380	0.0372	0.0386	0.0356
3	9–12	*R*	2.0	2.6530	2.8813	2.8725	2.8850	2.9110	2.8448
		t	0.051	0.0829	0.0744	0.0738	0.0736	0.0737	0.0719
4	13–16	*R*	2.0	2.1035	2.0071	1.9786	1.9927	1.9975	2.1732
		t	0.051	0.0535	0.0511	0.0512	0.0507	0.0509	0.0548
5	17–20	*R*	2.0	2.6784	2.8101	2.8328	2.8086	2.8239	2.7410
		t	0.051	0.0753	0.0717	0.0736	0.0716	0.0717	0.0697
6	21–24	*R*	2.0	2.1488	2.0703	2.0573	2.0656	2.0756	2.1600
		t	0.051	0.0547	0.0527	0.0533	0.0526	0.0529	0.0547
7	25–28	*R*	2.0	2.6238	2.6724	2.6071	2.6848	2.6672	2.6228
		t	0.051	0.0673	0.0680	0.0674	0.0685	0.0678	0.0666
8	29–32	*R*	2.0	2.1569	2.0965	2.0862	2.0884	2.0983	2.1834
		t	0.051	0.0549	0.0535	0.0540	0.0532	0.0534	0.0551
9	33–36	*R*	2.0	2.5038	2.5179	2.4670	2.4965	2.5103	2.5044
		t	0.051	0.0637	0.0642	0.0638	0.0637	0.0640	0.0635
10	37–40	*R*	2.0	2.1730	2.1475	2.1347	2.1583	2.1480	2.2406
		t	0.051	0.0553	0.0548	0.0533	0.0550	0.0547	0.0562
11	41–44	*R*	2.0	2.3748	2.3703	2.3654	2.4109	2.3618	2.4012
		t	0.051	0.0604	0.0605	0.0603	0.0615	0.0602	0.0603
12	45–48	*R*	2.0	1.9707	1.9847	1.9300	1.9938	1.9485	2.0082
		t	0.051	0.0502	0.0497	0.0501	0.0508	0.0496	0.0510
Weight, lb			69.11	111.20	108.80	109.22	108.35	108.70	108.30
No. of analyses			—	13	11	10	11	7	8

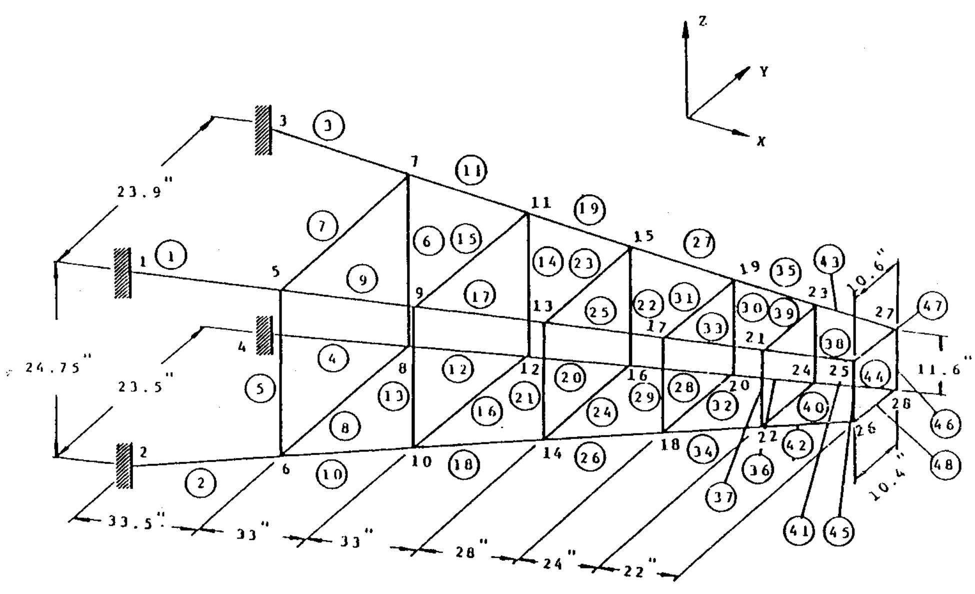

Fig. 5 Helicopter tail boom.

It can be deduced from Eqs. (18a), (18b), (19), (21), and (22) that

$$\frac{\partial l(\lambda)}{\partial \lambda_j} = g_j(Y), \qquad j = 1, 2, \ldots, \bar{J} \tag{23}$$

and from Eq. (20) that

$$\frac{\partial l(\lambda)}{\partial \lambda_j} = g_j(Y) = 0 \tag{24}$$

Substituting Eq. (18b) into Eq. (24), and combining Eq. (22), the iteration formula of λ can be given as

$$\lambda_p = [\nabla G^{(k)}(Y_p)^T H_f(Y_p)^{-1} \nabla G^{(k)}(Y_p)]^{-1}$$
$$\times [G^{(k)}(Y_p)^T - \nabla G^{(k)}(Y_p)^T H_f(Y_p)^{-1} \nabla f(Y_p)] \tag{25}$$

where $\nabla G(Y)$ is the gradient matrix of constraint functions; that is,

$$\nabla G(Y) = [\nabla g_1(Y), \nabla g_2(Y), \ldots, \nabla g_J(Y)]^T \tag{26}$$

Iteration procedures given in Eqs. (22) and (25) are repeated until the convergence is reached for the SQP problem given in Eq. (18). This iterative procedure ensures that the design constraints are satisfied after the dual problem is solved. After the solution Y^* and λ^* of approximate problem (18) are obtained, the new structural analysis is needed; then the $k+1$ stage approximation of Eq. (17) is constructed, and the process is repeated until the optimum solution is converged.

The flow chart in Fig. 1 shows the iteration scheme of the algorithm. The convergence is indicated by a sufficiently small relative objective function change in two successive iterations. Exact structural analyses are not needed for the SQP approximation problem; only spline interpolation is used to obtain the function values and

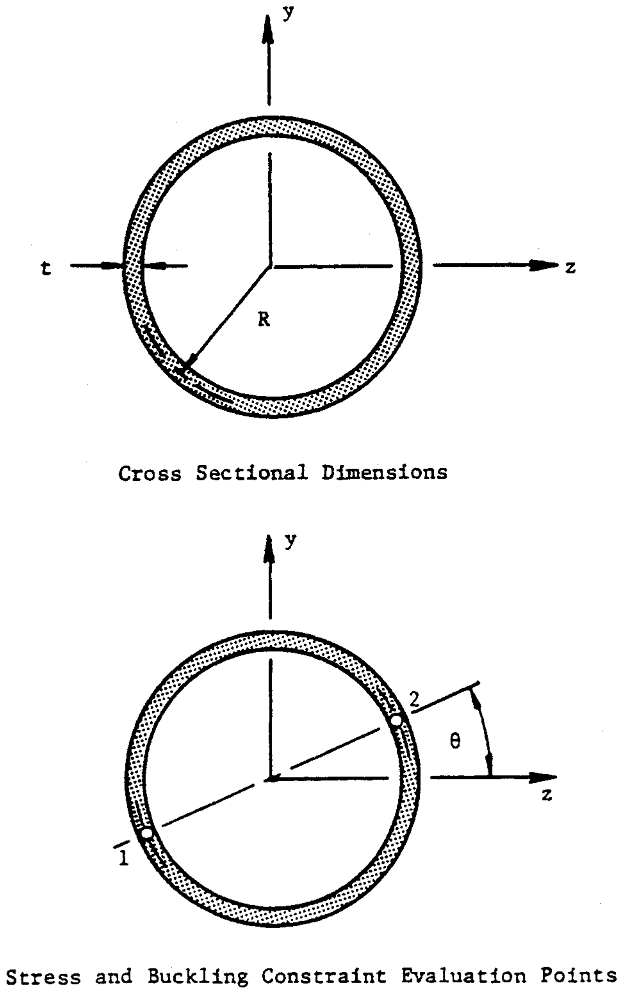

Fig. 6 Thin-walled tube cross section.

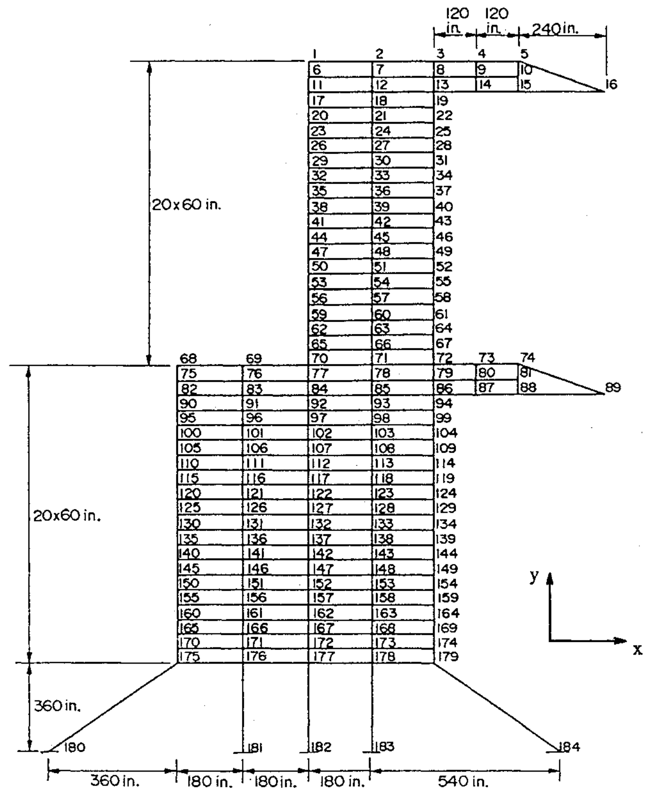

Fig. 7 313 frame member.

derivatives of constraint functions in each iteration. In general, 5–10 spline interpolations are needed to obtain the optimum solution of the SQP approximate problem. After the optimum solution of the SQP problem is obtained, an exact structural analysis is needed for updating the spline approximation of Eq. (17). In solving the complete optimization problem, fewer structural analyses are required because of the high accuracy of the multivariate spline approximation with adaptability.

Numerical Examples and Comparisons

Several frame examples are selected to examine the efficiency and robustness of the present optimization algorithm with multivariate spline approximation, and the results are compared with the previously published papers.[6–8] In Ref. 8, run 1, run 2, etc. represent different solution options, such as different design spaces and approximations, for each example. These examples include two member frame, 2 × 5 grillage, helicopter tail boom, and 313 member structures with multiple constraints subjected to sizes, stresses, displacements, and local buckling. In the first two examples, the frame members have thin-walled rectangular cross-sectional shapes; the third example has thin-walled tube cross-sectional members; and in the last example, *I*-section frame members are considered. During the optimization process, the cross-sectional dimensions are directly selected as design variables for the rectangular and tube-section frames, and the cross-sectional areas are taken as design variables for the *I*-section frame.

Example 1: Two Member Frame

A two member plane frame subject to a single out of plane load is shown in Fig. 2. The material properties and nodal loading are given as Young's modulus, $E = 20.74 \times 10^6$ N/cm^2; shear modulus,

Table 7 Iteration history for helicopter tail boom

Step no.	Ref. 8, run 1	Ref. 8, run 3	Ref. 8, run 4	Ref. 8, run 5	Present paper
1	69.11	69.11	69.11	69.11	69.043
2	95.86	93.97	97.87	105.44	98.699
3	107.94	105.45	108.65	112.34	113.430
4	110.69	109.59	109.62	110.20	109.456
5	109.51	109.52	108.74	109.15	108.052
6	110.13	109.45	108.34	108.90	108.082
7	109.33	109.40	108.74	108.70	108.031
8	109.25	109.36	108.66		108.031
9	108.83	109.27	108.48		
10	108.71	109.22	108.52		
11	108.80		108.35		

$G = 7.97 \times 10^6$ N/cm^2; Poisson's ratio, $\nu = 0.3$; mass density, $\rho = 2.77 \times 10^{-2}$ kg/cm^3, and allowable stress, $\sigma_a = 2.76 \times 10^4$ N/cm^2.

This structure is modeled using two thin-walled box section frame elements, each having four independent design variables: B, H, t_b, and t_h (Fig. 3). The constraints include sizes, stresses, and local buckling. The details on stress and buckling evaluation points and the initial design and side constraints are given in Ref. 8; the initial weight is 1220.78 kg. The final cross-sectional dimensions are listed in Table 1, and the iteration history is shown in Table 2. The convergence details of each iteration are listed in Table 3, which shows clearly the iteration procedure of the present optimization algorithm. The approximations constructed using spline functions reduced the constraint violations resulting from the dual method, and also, in several instances, further reduced the objective function. Typically, up to six iterations were needed using spline approximations. At the beginning of optimization, the dual problem had two variables for the active stress constraints. The nonlinearity index r for these two constraints has −1.97 and −1.58 values; therefore, the polynomial degree of spline functions with reciprocal variables was 2 for both active constraints. In the final

Table 8 Loads on 313 member frame

Load case	Node	x load, kips	y load, kips
1	15		−20.0
	16		−30.0
	88		−18.0
	89		−20.0
2	1	2.0	
	6–65 × 5	4.0	
	68, 75, 82	4.0	
	90–170 × 5	4.0	
3	5	−2.0	
	16, 19	−4.0	
	72–67 × 5	−4.0	
	74, 89	−4.0	
	94–174 × 5	−4.0	
4	15		−20.0
	16		−30.0
	88		−18.0
	89		−20.0
	1	2.0	
	6–65 × 5	4.0	
	68, 75, 82	4.0	
	90–170 × 5	4.0	
5	15		−20.0
	16		−30.0
	88		−18.0
	89		−20.0
	5	−2.0	
	16, 19	−4.0	
	72–67 × 5	−4.0	
	74, 89	−4.0	
	90–174 × 5	−4.0	

iteration, the dual problem still had two dual variables and the nonlinearity index r of both critical constraint functions was 1.

Example 2: 2 × 5 Grillage Structure

Figure 4 depicts a 2 × 5 grillage subject to a single-loading condition. The material properties and the nodal loading are given as Young's modulus, $E = 30.0 \times 10^6$ psi; shear modulus, $G = 11.5 \times 10^6$ psi; Poisson's ratio, $\nu = 0.2963$; allowable stress, $\sigma_a = 2.0 \times 10^4$ psi; displacement constraints in direction z of nodes 4, 7, and 10, -0.1 in. $\leq U \leq 1.0$ in.; applied loads: nodes 3, 6, 9, 12, and 15, $F_y = -9000$ lb; nodes 4, 7, 10, 13, and 16, $F_y = -3333$ lb; nodes 1 and 17, $M_z = -13{,}330$ in.-lb; nodes 2, 5, 8, 11, and 14, $M_x = 37{,}040$ in.-lb; and nodes 3, 6, 9, 12, and 15, $M_x = -27{,}780$ in.-lb; where F and M represent concentrated force and moment, respectively.

This structure is modeled with thin-walled box section frame elements (Fig. 3), each having four independent design variables (B, H, t_b, and t_h). Since both the structure and its loading are symmetric, one-half model is used in solving the design problem. This example has 16 members and 64 variables; only 16 independent design variables are considered by grouping members 1–6, 7–10, 11–14, and 15 and 16, as four sets. The grillage is designed for minimum material volume subject to constraints containing sizes, stresses, displacements, and local buckling. The initial design and side constraints are given in Ref. 8, and the initial volume is 32,393.4 in.^{3}. The final design of the cross-sectional dimensions is listed in Table 4, and the iteration history is shown in Table 5. During the optimization, the polynomial degrees of the spline functions varied from 2 to 1.

Example 3: Helicopter Tail Boom

The third example is a space frame idealization of a helicopter tail boom subject to single load condition (Fig. 5). The material properties and applied loads are given as Young's modulus, $E = 10.5 \times 10^6$ psi; shear modulus, $G = 40.4 \times 10^5$ psi; Possion's ratio, $\nu = 0.3$; weight density, $\rho = 0.1$ lb/in.^{3}; allowable stress, $\sigma_a = 4.2 \times 10^4$ psi; factor of safety (FS), FS = 1.25; displacement constraints in direction y and z of nodes 5–28, -0.5 in. $\leq U \leq 0.5$ in.; applied loads: nodes 13–16, $F_z = -140.0$ lb; node 25, $F_x = 1490.3$ lb, $F_y = 1691.8$ lb; node 26, $F_x = 1490.3$ lb, $F_y = -1365.8$ lb; node 27, $F_x = -1490.3$ lb, $F_y = 1691.8$ lb; and node 28, $F_x = -1490.3$ lb, $F_y = -1365.8$ lb.

All members of the structure are made of the same material and are modeled with thin-walled tube sections (Fig. 6) having two independent design variables (R and t). This example has 48 members and 96 variables; only 24 independent design variables are considered after variable linking by using 12 groups, each having members 1–4, 5–8, 9–12, . . . , 45–48, respectively. This structure is designed for minimum weight, subject to constraints on the nodal displacements in the y and z directions at nodes 5–28 and side constraints on element sizing variables. Stress constraints are also imposed at both ends of each member, along with column buckling and local wall buckling constraints (in the form of R/t ratio). The initial weight is 69.11 lb, and the design was started from a highly infeasible point. At the second iteration, the y-direction displacements of nodes 25 and 27 are active, and the corresponding nonlinearity indices are −2.88 and −1.58, respectively. At the optimum, the y-direction displacements of nodes 25 and 27 and the R/t constraints of members 1–4 and 9–48 are critical constraints. The nonlinearity indices of the displacements of nodes 25 and 27 are 1.005. The final design of the cross-sectional dimensions is listed in Table 6, and the iteration history is shown in Table 7. Even though eight iterations were needed for the present algorithm, the optimum weight was slightly lower than the reported values.

Example 4: 313 Member Frame

The frame structure shown in Fig. 7 has 313 members with an I section.[3] The cross-sectional area A is chosen as the primary variable with I_z and S_z expressed as explicit nonlinear functions of A in the form[3]: area moment of inertia, $I_z = 0.2072A^3$ and section modulus, $S_z = 0.3930A^2$.

This frame is subjected to five load conditions as given in Table 8. Stress and displacement constraints are considered. The material properties and constraints are given as Young's modulus, $E = 29 \times 10^6$ psi; Possion's ratio, $\nu = 0.3$; mass density, $\rho = 0.238$ lb/in.^{3}; allowable stress on each element, $\sigma_a = 29 \times 10^3$ psi; displacement constraints in x and y direction of all nodes, -12.0 in. $\leq U_x \leq 12.0$ in., -4.0 in. $\leq U_y \leq 4.0$ in.

The initial cross-sectional areas are 30.59 in.^{2} and the corresponding weight is 324,591.9 lb. The lower limit on the design variables was 0.12 in.^{2}. Figure 8 shows that the present algorithm converges in 10 iterations to a final weight of 123,137.1 lb. The compound scaling algorithm in Ref. 3 converges to 125,166.0 lb in 25 iterations. The design was started from a highly feasible point. At the end of second iteration several constraints were violated. Only the y-direction displacement constraint of node 16 under the first load case has two points of information, and the corresponding nonlinearity index is −1.005. At the optimum, 13 displacement constraints and 5 stress constraints are active, in which the y-direc-

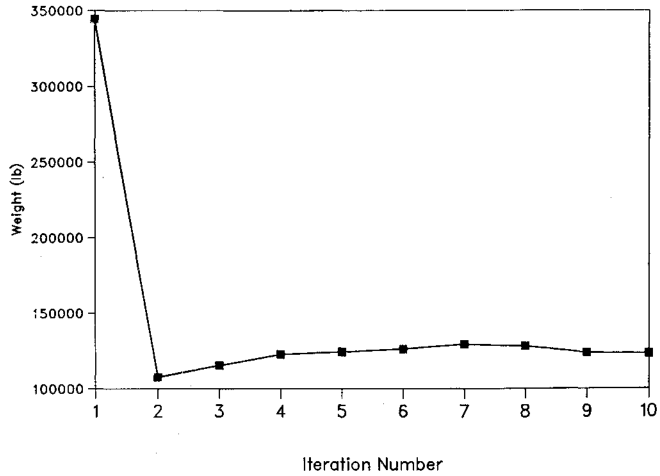

Fig. 8 Iteration history of 313 member frame.

tion displacement of node 16 under the fourth load case is the most critical and the corresponding nonlinearity index is 0.95.

Conclusions

An optimization algorithm based on multivariate spline approximation with adaptability is presented. The mathematical problem is solved using sequential quadratic programming and dual methods. Design reiterations using spline approximations reduced the constraint violations and objective function. The algorithm was demonstrated on frame structure designs with stress, displacement, and local buckling constraints. Design variables consisted of both detailed cross-sectional dimensions and relationships among area moment of inertia and cross-sectional areas. The algorithm constructed high quality approximations for both types of design variables and effectively solved the problems using 5–10 structural analyses. In the spline approximation, a maximum of five data points information was used so that the influence of far away points was ignored.

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